

On Hall's conjecture

Andrej Dujella

Abstract

We show that for any even positive integer δ there exist polynomials x and y with integer coefficients such that $\deg(x) = 2\delta$, $\deg(y) = 3\delta$ and $\deg(x^3 - y^2) = \delta + 5$.

Hall's conjecture asserts that for any $\varepsilon > 0$, there exists a constant $c(\varepsilon) > 0$ such that if x and y are positive integers satisfying $x^3 - y^2 \neq 0$, then $|x^3 - y^2| > c(\varepsilon)x^{1/2-\varepsilon}$. It is known that Hall's conjecture follows from the *abc*-conjecture. For a stronger version of Hall's conjecture which is equivalent to the *abc*-conjecture see [3, Ch. 12.5]. Originally, Hall [8] conjectured that there is $C > 0$ such that $|x^3 - y^2| \geq C\sqrt{x}$ for positive integers x, y with $x^3 - y^2 \neq 0$, but this formulation is unlikely to be true. Danilov [4] proved that $0 < |x^3 - y^2| < 0.97\sqrt{x}$ has infinitely many solutions in positive integers x, y ; here 0.97 comes from $54\sqrt{5}/125$. For examples with “very small” quotients $|x^3 - y^2|/\sqrt{x}$, up to 0.021 , see [7] and [9].

It is well known that for non-constant complex polynomials x and y , such that $x^3 \neq y^2$, we have $\deg(x^3 - y^2)/\deg(x) > 1/2$. More precisely, Davenport [6] proved that for such polynomials the inequality

$$\deg(x^3 - y^2) \geq \frac{1}{2} \deg(x) + 1 \quad (1)$$

holds. This statement also follows from Stothers-Mason's *abc* theorem for polynomials (see, e.g., [10, Ch. 4.7]). Zannier [12] proved that for any positive integer δ there exist complex polynomials x and y such that $\deg(x) = 2\delta$, $\deg(y) = 3\delta$ and x, y satisfy the equality in Davenport's bound (1). In his previous paper [11], he related the existence of such examples with coverings of the Riemann sphere, unramified except above 0 , 1 and ∞ .

⁰ 2000 Mathematics Subject Classification: 11C08, 11D25, 11D75.

Key words: Hall's conjecture, integer polynomials.

The author was supported by the Ministry of Science, Education and Sports, Republic of Croatia, grant 037-0372781-2821.

It is natural to ask whether examples with the equality in (1) exist for polynomials with integer (rational) coefficients. Such examples are known only for $\delta = 1, 2, 3, 4, 5$ (see [1, 7]). The first example for $\delta = 5$ was found by Birch, Chowla, Hall and Schinzel [2]. It is given by

$$x = \frac{t}{9}(t^9 + 6t^6 + 15t^3 + 12), \quad y = \frac{1}{54}(2t^{15} + 18t^{12} + 72t^9 + 144t^6 + 135t^3 + 27),$$

while then

$$x^3 - y^2 = -\frac{1}{108}(3t^6 + 14t^3 + 27)$$

(note that x, y are integers for $t \equiv 3 \pmod{6}$). One more example for $\delta = 5$ has been found by Elkies [7]:

$$\begin{aligned} x &= t^{10} - 2t^9 + 33t^8 - 12t^7 + 378t^6 + 336t^5 + 2862t^4 + 2652t^3 + 14397t^2 + 9922t + 18553, \\ y &= t^{15} - 3t^{14} + 51t^{13} - 67t^{12} + 969t^{11} + 33t^{10} + 10963t^9 + 9729t^8 + 96507t^7 \\ &\quad + 108631t^6 + 580785t^5 + 700503t^4 + 2102099t^3 + 1877667t^2 + 3904161t + 1164691, \\ x^3 - y^2 &= 4591650240t^6 - 5509980288t^5 + 101934635328t^4 + 58773123072t^3 \\ &\quad + 730072388160t^2 + 1151585880192t + 5029693672896. \end{aligned}$$

In these examples we have

$$\deg(x^3 - y^2)/\deg(x) = 0.6,$$

and it seems that no examples of polynomials with integer coefficients, satisfying $x^3 - y^2 \neq 0$ and $\deg(x^3 - y^2)/\deg(x) < 0.6$, were published until now.

In this note we will show the following result.

Theorem 1 *For any $\varepsilon > 0$ there exist polynomials x and y with integer coefficients such that $x^3 \neq y^2$ and $\deg(x^3 - y^2)/\deg(x) < 1/2 + \varepsilon$.*

More precisely, for any even positive integer δ there exist polynomials x and y with integer coefficients such that $\deg(x) = 2\delta$, $\deg(y) = 3\delta$ and $\deg(x^3 - y^2) = \delta + 5$.

As an immediate corollary we obtain a nontrivial lower bound for the number of integer solutions to the inequality $|x^3 - y^2| < x^{1/2+\varepsilon}$ with $1 \leq x \leq N$ (heuristically, it is expected that this number is around N^ε).

Corollary 1 *For any $\varepsilon > 0$ and positive integer N by $\mathcal{S}(\varepsilon, N)$ we denote the number of integers x , $1 \leq x \leq N$, for which there exists an integer y such that $0 < |x^3 - y^2| < x^{1/2+\varepsilon}$. Then we have*

$$\mathcal{S}(\varepsilon, N) \gg N^{\varepsilon/(5+4\varepsilon)}.$$

Indeed, take δ to be the smallest even integer greater than $5/(2\varepsilon)$, so that $5/(2\varepsilon) < \delta < 5/(2\varepsilon) + 2$, and take $x = x(t)$, $y = y(t)$ as in Theorem 1. Then for sufficiently large t we have $x = O(t^{2\delta})$ and $|x^3 - y^2| = O(t^{\delta+5}) = O(x^{\frac{1}{2} + \frac{5}{2\delta}}) < x^{1/2+\varepsilon}$. Therefore,

$$\mathcal{S}(\varepsilon, N) \gg N^{1/(2\delta)} \gg N^{\varepsilon/(5+4\varepsilon)}.$$

Here is an explicit example which improves the quotient $\deg(x^3 - y^2)/\deg(x) = 0.6$ from the above mentioned examples by Birch, Chowla, Hall, Schinzel and Elkies, as $\deg(x^3 - y^2)/\deg(x) = 31/52 = 0.5961\dots$:

$$\begin{aligned} x = & 281474976710656t^{52} + 3799912185593856t^{50} + 24189255811072000t^{48} + 96537120918732800t^{46} \\ & + 270892177293312000t^{44} + 568175382432317440t^{42} + 924393098014883840t^{40} \\ & + 1194971570896896000t^{38} + 1247222961904025600t^{36} + 1062249296822272000t^{34} \\ & + 743181990714408960t^{32} + 428630517911388160t^{30} + 203971125837824000t^{28} + 100663296t^{27} \\ & + 79960271015116800t^{26} + 729808896t^{25} + 25720746147840000t^{24} + 2359296000t^{23} \\ & + 6745085391667200t^{22} + 4482662400t^{21} + 1428736897843200t^{20} + 5554176000t^{19} \\ & + 241375027200000t^{18} + 4706795520t^{17} + 31982191104000t^{16} + 2782494720t^{15} + 3250264320000t^{14} \\ & + 1148928000t^{13} + 245895686400t^{12} + 326476800t^{11} + 13292822400t^{10} + 61776000t^9 + 484380000t^8 \\ & + 7344480t^7 + 10894000t^6 + 496080t^5 + 130625t^4 + 15750t^3 + 629t^2 + 150t + 4, \\ y = & 4722366482869645213696t^{78} + 95627921278110315577344t^{76} + 931486788746037518401536t^{74} \\ & + 5812273909720700361375744t^{72} + 26102714713365300532740096t^{70} + 89873242715073754863501312t^{68} \\ & + 246761827996223603178733568t^{66} + 554869751478978106456276992t^{64} \\ & + 1041377162422256031202541568t^{62} + 1654256777803799676753805312t^{60} \\ & + 2247766244734980591395536896t^{58} + 2633529391786763986554322944t^{56} \\ & + 2676840149412734907329806336t^{54} + 2533274790395904t^{53} + 2371433108159248512627769344t^{52} \\ & + 35465847065542656t^{51} + 1837294956807449113993936896t^{50} + 234486247786020864t^{49} \\ & + 1247823926411289395000770560t^{48} + 973569167884025856t^{47} + 743994544482135039635619840t^{46} \\ & + 2847272221544546304t^{45} + 389682593956278112836648960t^{44} + 6236328797675716608t^{43} \\ & + 179279686440609529032867840t^{42} + 10618254681610125312t^{41} + 72388134028773255869890560t^{40} \\ & + 14399046085119049728t^{39} + 25611943886548098204303360t^{38} + 15806610071787405312t^{37} \\ & + 7922395450159324505047040t^{36} + 14200560742834372608t^{35} + 2135839807968003238133760t^{34} \\ & + 10514148446410113024t^{33} + 499883693495498613719040t^{32} + 6441026076788391936t^{31} \\ & + 101073262762096181903360t^{30} + 3269189665642512384t^{29} + 17550157782838363029504t^{28} \\ & + 1373442845007937536t^{27} + 2598168579136061177856t^{26} + 476068223096193024t^{25} \\ & + 325093317533140516864t^{24} + 135395930768670720t^{23} + 34019036843474681856t^{22} \\ & + 31339645700014080t^{21} + 2939255644452962304t^{20} + 5838612910571520t^{19} + 206402445920944128t^{18} \\ & + 862650209710080t^{17} + 11551766627438592t^{16} + 99129281310720t^{15} + 502656091170048t^{14} \\ & + 8633278321920t^{13} + 16468534726592t^{12} + 550276346880t^{11} + 389483950128t^{10} + 24450210720t^9 \\ & + 6312333144t^8 + 705350880t^7 + 68685241t^6 + 11812545t^5 + 642429t^4 + 94050t^3 + 6591t^2 + 225t + 19, \end{aligned}$$

$$\begin{aligned}
x^3 - y^2 = & -905969664t^{31} - 8380219392t^{29} - 35276193792t^{27} - 89379569664t^{25} - 151909171200t^{23} \\
& - 182680289280t^{21} - 159752355840t^{19} - 102786416640t^{17} - 48661447680t^{15} - 16772918400t^{13} \\
& - 4116359520t^{11} - 692649360t^9 - 75171510t^7 - 297t^6 - 4749570t^5 - 891t^4 - 144450t^3 - 891t^2 \\
& - 1350t - 297.
\end{aligned}$$

Now we describe the general construction. Let us define the binary recursive sequence by

$$a_1 = 0, \quad a_2 = t^2 + 1, \quad a_m = 2ta_{m-1} + a_{m-2}.$$

Thus, for $m \geq 2$, a_m is a polynomial in variable t , of degree m . Put $u = a_{k-1}$ and $v = a_k$ for an odd positive integer $k \geq 3$. We search for examples with $x = O(v^2)$, $y = O(v^3)$ and $x^3 - y^2 = O(v)$. Note that

$$v^2 - 2tuv - u^2 = -(a_2^2 - 2ta_1a_2 - a_1^2) = -(t^2 + 1)^2. \quad (2)$$

Therefore, we may take

$$\begin{aligned}
x &= av^2 + buv + cu + dv + e, \\
y &= fv^3 + gv^2u + hv^2 + iuv + ju + mv + n,
\end{aligned}$$

with unknown coefficients a, b, c, \dots, n , which will be determined so that in the expression for $x^3 - y^2$ the coefficients with $v^6, uv^5, v^5, \dots, v^2, uv$ are equal to 0. We find the following (polynomial) solution:

$$\begin{aligned}
x &= v^2 - 2tuv + 6v - 6tu + (t^4 + 5t^2 + 4), \\
y &= -2tv^3 + (4t^2 + 1)uv^2 - 9tv^2 + (18t^2 + 9)uv + (-2t^5 - 4t^3 - 2t)v \\
&\quad + (t^4 + 20t^2 + 19)u + (-9t^5 - 18t^3 - 9t).
\end{aligned}$$

Using (2), it is easy to check that we have

$$x^3 - y^2 = -27(t^2 + 1)^2(2v - 2tu + 11t^2 + 11).$$

Therefore, $\deg(x) = 2k - 2$ and $\deg(x^3 - y^2) = k + 4$. Also,

$$\deg(x^3 - y^2) / \deg(x) = (k + 4) / (2k - 2),$$

which tends to $1/2$ when k tends to infinity. The above explicit example corresponds to $k = 27$.

Comparing with Davenport's bound, our polynomial x and y satisfy

$$\deg(x^3 - y^2) = \frac{1}{2} \deg(x) + 5.$$

Thus, although our examples (x, y) do not give the equality in Davenport's bound (1), they are very close to the best possible result for $\deg(x^3 - y^2)$, and it seems that this is the first known result of the form that $\deg(x^3 - y^2) - \frac{1}{2} \deg(x)$ is bounded by an absolute constant, for polynomials x, y with integer coefficients and arbitrarily large degrees.

Since $(t^2 + 1)$ divides a_m for all m , it could be noted that $(t^2 + 1)$ divides x and $(t^2 + 1)^2$ divides y . Hence, with $x = (t^2 + 1)X$ and $y = (t^2 + 1)^2Y$, we have

$$\deg(X^3 - (t^2 + 1)Y^2) = \frac{1}{2} \deg(X).$$

This shows that the only branch points of the rational function x^3/y^2 are 0, 1 and ∞ , which is in agreement with the results of Zannier [11, 12].

Let us give an interpretation of our result in terms of polynomial Pell's equations. Following a suggestion by N. Elkies, we put $v - tu = (t^2 + 1)z$. Then the expressions of x and $x^3 - y^2$ simplify considerably, and we get $x = (t^2 + 1)(z^2 + 6z + 4)$, $x^3 - y^2 = -27(t^2 + 1)^3(2z + 11)$ which gives $y^2 = (t^2 + 1)^3(z^2 + 1)(z^2 + 9z + 19)^2$. Thus, we need that $z^2 + 1 = (t^2 + 1)w^2$, i.e

$$z^2 - (t^2 + 1)w^2 = -1. \quad (3)$$

The fundamental solution of Pell's equation (3) is $(z, w) = (t, 1)$. Taking $t = z$, we obtain the identity

$$(z^2 + 6z + 4)^3 - (z^2 + 1)(z^2 + 9z + 19)^2 = -27(2z + 11),$$

which is equivalent to Danilov's example [4] (and by taking $z^2 + 1 = 5w^2$ and $2z + 11 \equiv 0 \pmod{125}$, we get a well-known sequence of numerical examples with $|x^3 - y^2| < \sqrt{x}$).

However, if we consider (3) as a polynomial Pell's equation (in variable t), we obtain the sequence of solutions

$$z_1 = t, \quad z_2 = 4t^3 + 3t, \quad z_k = (4t^2 + 2)z_{k-1} - z_{k-2}.$$

This gives exactly the sequences of polynomials x and y , as given above.

Remark 1 In [5], Danilov considered small values of $|x^4 - Ay^2|$, for integers A satisfying certain conditions. Using the formula

$$(27z + 7)^4 - (81z + 20)^2 \cdot \frac{(81z + 22)^2 + 2}{81} = 4z + 1, \quad (4)$$

he proved that if the Pellian equation $u^2 - 81Av^2 = -2$ has a solution, then the inequality $|x^4 - Ay^2| < \frac{4}{27}|x|$ has infinitely many integer solutions x, y . By applying a similar construction, as above, to Danilov's formula (4), we obtain the sequences x_k and y_k of polynomials in variable t with $\deg(x_k) = 2k + 1$, $\deg(y_k) = 4k$ and $\deg(x^4 - (t^2 + 2)y^2) = \deg(x) = 2k + 1$. For example, for $k = 3$ we have

$$\begin{aligned} x &= 8t^7 + 28t^5 + 28t^3 + 7t - 1, \\ y &= 64t^{13} + 384t^{11} + 880t^9 + 960t^7 - 16t^6 + 504t^5 - 40t^4 + 112t^3 - 24t^2 + 7t - 2, \end{aligned}$$

and then

$$x^4 - (t^2 + 2)y^2 = 32t^7 + 112t^5 + 112t^3 + 28t - 7.$$

Acknowledgements. The author is grateful to Yann Bugeaud, Noam Elkies, Clemens Fuchs, Boris Širola and Umberto Zannier for their very interesting and useful comments on the previous version of this note.

References

- [1] F. Beukers, C. L. Stewart, *Neighboring powers*, J. Number Theory **130** (2010), 660–679.
- [2] B. J. Birch, S. Chowla, M. Hall, Jr., A. Schinzel, *On the difference $x^3 - y^2$* , Norske Vid. Selsk. Forh. (Trondheim) **38** (1965), 65–69.
- [3] E. Bombieri, W. Gubler, *Heights in Diophantine geometry*, Cambridge University Press, Cambridge, 2006.
- [4] L. V. Danilov, *The Diophantine equation $x^3 - y^2 = k$ and Hall's conjecture*, Math. Notes Acad. Sci. USSR **32** (1982), 617–618.
- [5] L. V. Danilov, *The Diophantine equations $x^m - Ay^n = k$* , Math. Notes Acad. Sci. USSR **46** (1989), 914–919.
- [6] H. Davenport, *On $f^3(t) - g^2(t)$* , Norske Vid. Selsk. Forh. (Trondheim) **38** (1965), 86–87.
- [7] N. Elkies, *Rational points near curves and small nonzero $|x^3 - y^2|$ via lattice reduction*, Lecture Notes in Computer Science **1838** (proceedings of ANTS-4, 2000; W. Bosma, ed.), 33–63.
- [8] M. Hall, Jr., *The Diophantine equation $x^3 - y^2 = k$* , in: Computers in Number Theory, A. O. L. Atkin and B. J. Birch (eds.), Proc. Oxford 1969, Academic Press, 1971, 173–198.

- [9] I. Jiménez Calvo, J. Herranz, G. Sáez, *A new algorithm to search for small nonzero $|x^3 - y^2|$ values*, Math. Comp. **78** (2009), 2435–2444.
- [10] S. Lang, Algebra, 3rd ed., Addison-Wesley, Reading, 1993.
- [11] U. Zannier, *Some remarks on the S -unit equation in function fields*, Acta Arith. **64** (1993), 87–98.
- [12] U. Zannier, *On Davenport's bound for the degree of $f^3 - g^2$ and Riemann's Existence Theorem*, Acta Arith. **71** (1995), 103–137.

Department of Mathematics, University of Zagreb,
 Bijenička cesta 30, 10000 Zagreb, Croatia
E-mail address: duje@math.hr